BRL

REPORT No. 898

Notes on a Theory of Spinning Shell

R. H. KENT

DEPARTMENT OF THE ARMY PROJECT No. 503-03-001
ORDNANCE RESEARCH AND DEVELOPMENT PROJECT No. TB3-0108

BALLISTIC RESEARCH LABORATORIES

ABERDEEN PROVING GROUND, MARYLAND

ERRATA SHEET FOR BRL REPORT NO. 898

Notes on a Theory of Spinning Shell

by

R. H. Kent

Page 2: 12th line from bottom - add ** on right side of μ.

Page 2: Second footnote, in the equation - replace d^2 by d^3.

Page 14: Footnote - replace u^2 by v^2.

Page 31: First equation - replace u by v.

Page 32 - 3rd line from top - replace u by v.

b. "Initial distribution has been made of this report in accordance with the distribution list contained herein. Additional distribution without recourse to the Ordnance Office may be made to United States military organizations, and to such of their contractors as they certify to be cleared to receive this report and to need it in the furtherance of a military contract."

BALLISTIC RESEARCH LABORATORIES

REPORT NO. 898

February 1954

NOTES ON A THEORY OF SPINNING SHELL

R. H. Kent

Department of the Army Project No. 503-03-001
Ordnance Research and Development Project No. TB3-0108

ABERDEEN PROVING GROUND, MARYLAND

To

R. H. Fowler

TABLE OF CONTENTS

BALLISTIC RESEARCH LABORATORIES

REPORT NO. 898

RHKent/8281/iph
Aberdeen Proving Ground, Md.
5 February 1954

NOTES ON A THEORY OF SPINNING SHELL

ABSTRACT

The theory expounded here is based mainly on that of Fowler, Gallop, Lock and Richmond* except that the coordinate system is attached to the actual trajectory instead of the particle trajectory. They are followed in their fortunate choice of symbols, especially for the damping factors.

The equations for the complex yaw are developed on the usual linear assumption and their solutions obtained. The dynamical stability of shell is discussed. It is here pointed out that, for practical purposes, it is not essential that a shell be dynamically stable at all points of its trajectory. A reassuring comparison is made between the results of this theory and those of FGLR and Kelley and McShane.

*Phil. Trans. Roy. Soc. A, vol. 221, pp. 295-387 (1920).

DEFINITION OF SYMBOLS

a. Dynamical constants and variables.
b. Geometric magnitudes and vectors.
c. Aerodynamic forces.
d. Damping factors.

a. Dynamical constants and variables:

		Units
A	Axial moment of inertia of the shell	$m\ell^2$
B	Transverse moment of inertia	$m\ell^2$
m	Mass	m
v	Velocity	ℓ/t
N	Spin of the shell about its longitudinal axis	$1/t$

b. Geometric magnitudes and vectors:

i, j and k - unit vectors pointing along the directions of the x, y and z axes respectively.

A the unit vector having the direction of the axis of the shell. If the direction cosines of A are ℓ, m and n, $A = \ell i + mj + nk$.

δ the angle of yaw - the angle between the trajectory and the longitudinal axis of the shell.

w $A \times \dot{A}$, the vector angular velocity of the axis of the shell.

c. Aerodynamic forces:

		Units
D	Drag - the force acting on the projectile along the trajectory opposite to its direction of motion.	$m\ell/t^2$
M	Overturning moment assumed to act in the plane of the yaw and assumed to be proportional to sin δ. Thus, M may be written as $M = \mu \sin \delta$, where μ is called the moment co-factor.	$m\ell^2/t^2$
L	Cross wind force - a force acting perpendicularly to the trajectory, in the plane of the yaw; assumed to be proportional to sin δ.	$m\ell/t^2$
Hw	Yawing moment due to yawing - when a shell yaws, there is a torque exerted on it, the axis of which coincides with the axis of the yawing motion and exerts a moment which opposes the yawing motion. This moment is called the yawing moment due to yawing and is represented by Hw.	$m\ell^2/t^2$

		Units
K	Magnus force - corresponding to the cross wind force, when a shell spins, there is a Magnus force acting on it, proportional to sin δ, but acting perpendicularly to the plane of the yaw.	$m\ell/t^2$
J	Magnus moment - the moment of the Magnus force.	$m\ell^2/t^2$
$-AN\Gamma$	Rolling moment - the moment that opposes the spin of the shell.	$m\ell^2/t^2$

d. Damping factors*:

ℓ The rolling moment damping factor

$$\ell = \frac{A\dot{N}}{AN}$$

κ The cross wind force damping factor

$$\kappa = \frac{L}{mv \sin \delta}$$

h The yawing moment due to yawing damping factor

$$h = -H/B$$

λ The Magnus force damping factor

$$\lambda = \frac{K}{mv\ N \sin \delta}$$

γ The Magnus moment damping factor

$$\gamma = \frac{J}{AN \sin \delta} .$$

---

*These damping factors all have 1/t for unit.

Chapter I

Motion with Simplified Force System.

To simplify the problem and to bring out the essential points of the
theory, we shall consider first a shell subject to a force system con-
M sisting only of the overturning moment, M, the drag, D, and a spin destroy-
D ing couple or rolling moment. We take

$$M = \mu \sin \delta$$

μ where μ is the moment co-factor and δ is the angle of yaw. We assume the
δ magnitude of the rolling moment to be $AN\Gamma$ where A is the axial moment of
A inertia of the shell and N the spin. The axis of the torque $AN\Gamma$ is that
N of the shell.
Γ

Gravity, the cross wind force, yawing moment due to yawing, etc., are for the present omitted.

Since there are no forces transverse to the trajectory, the trajectory
is linear. Therefore a coordinate system with origin at the center of
x gravity of the shell, with its x axis pointing along the trajectory, with
y its y axis in a vertical plane, and the z axis pointing to the right, while
z not Galilean with respect to translation, is Galilean so far as angular
motions are concerned. See Figure 1.

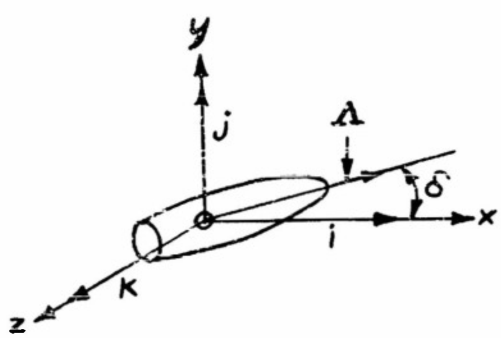

Figure 1

i The unit vectors having the directions of the x, y, z axes are designated
j as **i**, **j**, **k**, respectively. (These vectors as well as all other vectors
k are written with heavy letters.) We define the relevant quantities.

B B = moment of inertia about a transverse axis through the center of gravity.

A **A** = unit vector in the direction of the axis of the shell.

w **w** = the resultant vector angular velocity of the longitudinal axis about a transverse axis through the center of gravity (also called the cross spin).

H **H** = total vector angular momentum of shell with respect to O.

By reason of the symmetry of revolution which the shell is assumed to possess, the total angular momentum is the sum of two components:

(a) the component about the axis, which has the direction A and is represented by the vector ANA, and

(b) the component transverse to the axis, which is represented by the vector Bw.

The vector cross spin **w** is equal in magnitude to $\dot{A}$* and is perpendicular to A and $\dot{A}$. We thus have

$$\mathbf{w} = A \times \dot{A}.$$

The total angular momentum,

$$H = ANA + B\left[A \times \dot{A}\right].$$

The vector equation of motion is that the rate of change of angular

G momentum, $\dot{H}$, is equal to the impressed vector couple, designated by G.

If we take N as variable we find that

$$\dot{H} = A\dot{N}A + AN\dot{A} + B\left[\dot{A} \times \dot{A}\right] + B\left[A \times \ddot{A}\right].$$

The $\dot{A} \times \dot{A}$ term is zero since $\dot{A}$ and $\dot{A}$ are parallel. Thus

(1) $$\dot{H} = A\dot{N}A + AN\dot{A} + B\left[A \times \ddot{A}\right] = G.$$

On the present assumption, the vector couple G consists of two terms, the couple M and rolling moment $AN\Gamma$. Let us see how we may express them as vectors.

M is equal in magnitude to μ** sin δ and the axis of the couple is obviously perpendicular to the trajectory and hence to the unit vector **i**, and to the axis of the shell.

Consider the vector, $\left[i \times A\right]$; this vector is, by definition of the vector product, perpendicular to **i** and A and has the magnitude sin δ. Hence, the moment M is expressed as the vector, $\mu\left[i \times A\right]$. The rolling moment, $AN\Gamma$, has by definition A as its axis and the moment is assumed to be negative. Hence we represent it vectorially as $-AN\Gamma A$.

Replacing G in the preceding equation by its two components, $\mu\left[i \times A\right]$ and $-AN\Gamma A$, the equation becomes

(1a) $$\dot{H} = A\dot{N}A + AN\dot{A} + B\left[A \times \ddot{A}\right] = \mu\left[i \times A\right] - AN\Gamma A;$$

this is the vector equation of motion.

* Differentiation with respect to the time, t, is indicated by the superposed dot.

**In ballistic notation, $\mu = K_M \rho d^2 v^2$.

It should be noted that since A is a unit vector, $\dot{A}$ is necessarily perpendicular to A, while $[A \times \ddot{A}]$ and $[i \times A]$ are perpendicular to A by the definition of a vector product. Hence, if we take the scalar product of A into equation (1), all the terms vanish except $A\dot{N}(A \cdot A)$ and $-AN\Gamma\ (A \cdot A)$. From this

$$A\dot{N} = -AN\Gamma$$

and

$$A\dot{N}A = -AN\Gamma A.$$

By virtue of this, it is obvious that (1a) may be written

(1.01)
$$AN\dot{A} + B\left[A \times \ddot{A}\right] = \mu\left[i \times A\right] .$$

If the direction cosines of A with respect to the x, y, z axes are ℓ, m, and n,

$$A = \ell i + mj + nk$$

$$\dot{A} = \dot{\ell} i + \dot{m}j + \dot{n}k$$

$$\ddot{A} = \ddot{\ell} i + \ddot{m}j + \ddot{n}k$$

and

$$\left[A \times \ddot{A}\right] = (\ell i + mj + nk) \times (\ddot{\ell} i + \ddot{m}j + \ddot{n}k) .$$

Upon performing the vector multiplication, remembering that

$i \times i = 0$, $i \times j = k$, $j \times i = -k$, etc., we have

$$A \times \ddot{A} = (m\ddot{n} - n\ddot{m})\,i + (n\ddot{\ell} - \ell\ddot{n})j + (\ell\ddot{m} - m\ddot{\ell})k .$$

For $i \times A$, we have

$$i \times (\ell i + mj + nk) = -\,nj + mk .$$

Equation (1.01) written out in full becomes

(1.011)
$$AN(\dot{\ell} i + \dot{m}j + \dot{n}k) + B\Big[(m\ddot{n} - n\ddot{m})i + (n\ddot{\ell} - \ell\ddot{n})j + {}$$
$${} + (\ell\ddot{m} - m\ddot{\ell})k\Big] = \mu\left[-\,nj + mk\right] .$$

In this theory, the assumption is made that the yaw is small. If the yaw is small, then the i component, the magnitude of which is cos δ, is always taken as unity, and therefore the i component is not involved in this theory.

Since the i, j and k components are independent, there are three equations, one for each component. We take the j and k components:

(2) (j component) $AN\dot{n} + B\,(n\ddot{\ell} - \ell\ddot{n}) = -\,\mu n,$

(3) (k component) $AN\dot{m} + B\,(\ell\ddot{m} - m\ddot{\ell}) = +\,\mu m .$

ℓ is equal in magnitude to $\cos\delta \approx 1 - \frac{\delta^2}{2}$ and $\ddot{\ell} \approx -\dot{\delta}^2 - \delta\ddot{\delta}$.

Now, m and n are of the order of δ. Hence, in neglecting such terms as $n\ddot{\ell}$, we are neglecting a term of the order of $(\delta\dot{\delta}^2 - \delta^2\ddot{\delta})$ in comparison with δ and in taking $\ell = 1$, we are neglecting a term of the order of δ^2 in comparison with unity. We now take $\ell = 1$ and neglect the terms in $\ddot{\ell}$.

Equations (2) and (3) then become

(2.1) $$AN\dot{m} - B\ddot{n} = -\mu n$$

(3.1) $$AN\dot{n} + B\ddot{m} = +\mu m\,.$$

Multiply (2.1) by $i = \sqrt{-1}$ and then subtract from (3.1) with the result,

$$AN(-i\dot{m} + \dot{n}) + B(\ddot{m} + i\ddot{n}) = \mu\,(m + in).$$

Since $-(i)^2 = -1(-1) = +1$, the first term may be written $-iAN(\dot{m} + i\dot{n})$ and the equation becomes

(4) $$-iAN(\dot{m} + i\dot{n}) + B(\ddot{m} + i\ddot{n}) = \mu\,(m + in).$$

η We now substitute η for $(m + in)$ and call η the complex yaw.

The complex yaw η is represented on a plane perpendicular to the trajectory as indicated in Figure 2.

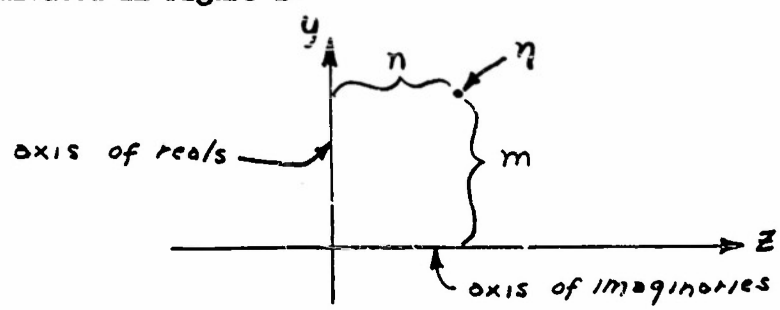

Figure 2

The axis of reals points upward in a vertical plane along the y axis and the axis of imaginaries points to the right along the z axis.

With η substituted for $m + in$, (4) becomes, upon division by B

(4.1) $$\ddot{\eta} - i\,\Omega\,\dot{\eta} - \frac{\mu}{B}\,\eta = 0,$$

Ω if Ω is used to represent AN/B.

To solve this equation, it is convenient first to eliminate the $\dot{\eta}$ term. Let us write it for future convenience in the form

$$\ddot{\eta} - iE\dot{\eta} - F\eta = 0 . \tag{4.2}$$

We now make the substitution

$$\eta = y \exp\left[\frac{1}{2} i \int_0^t E dt\right] .$$

Upon differentiating once, we have

$$\dot{\eta} = \dot{y} \exp\left[\frac{1}{2} i \int_0^t E dt\right] + y \frac{d}{dt}\left\{\exp\left[\frac{1}{2} i \int_0^t E dt\right]\right\} =$$

$$= \dot{y} \exp\left[\frac{1}{2} i \int_0^t E dt\right] + y \exp\left[\frac{1}{2} i \int_0^t E dt\right] \left(\frac{1}{2} i E\right) =$$

$$= \exp\left[\frac{1}{2} i \int_0^t E dt\right] \left[\dot{y} + \frac{1}{2} i E y\right] .$$

The second differentiation gives:

$$\ddot{\eta} = \frac{d}{dt}\left\{\exp\left[\frac{1}{2} i \int_0^t E dt\right]\right\} \left(\dot{y} + \frac{1}{2} iEy\right) + \exp\left[\frac{1}{2} i \int_0^t E dt\right] \frac{d}{dt}\left(\dot{y} + \frac{iEy}{2}\right) =$$

$$= \exp\left[\frac{1}{2} i \int_0^t E dt\right] \left[\frac{1}{2} i E\left(\dot{y} + \frac{1}{2} iEy\right) + \ddot{y} + \frac{iE\dot{y}}{2} + \frac{i\dot{E}y}{2}\right] =$$

$$= \exp\left[\frac{1}{2} i \int_0^t E dt\right] \left[\ddot{y} + \left(\frac{1}{2} iE + \frac{1}{2} iE\right)\dot{y} + \left(-\frac{1}{4} E^2 + \frac{i\dot{E}}{2}\right)y\right] .$$

Upon substituting for η, $\dot{\eta}$, and $\ddot{\eta}$ in (4.2) it becomes

$$\exp\left[\frac{1}{2} i \int_0^t E dt\right] \left\{\left[\ddot{y} + iE\dot{y} + \left(-\frac{1}{4} E^2 + \frac{i\dot{E}}{2}\right)y\right] - iE\left(\dot{y} + \frac{1}{2} iEy\right) - Fy\right\} =$$

$$= \exp\left[\frac{1}{2} i \int_0^t E dt\right] \cdot \left[\ddot{y} + \left(\frac{E^2}{4} - F + \frac{1}{2} i\dot{E}\right)y\right] = 0$$

and the equation for y is

$$\ddot{y} + \left(\frac{E^2}{4} - F + \frac{1}{2} i \dot{E}\right)y = 0 . \tag{4.3}$$

Upon substituting for E and F their values as obtained from (4.1) and for the present taking Ω as constant, that is, neglecting E in (4.3), we have

(4.4) $$\ddot{y} + \left(\frac{\Omega^2}{4} - \frac{\mu}{B}\right)y = 0 .$$

This may be written

$$\ddot{y} + \left(\frac{\Omega}{2}\right)^2 \left[1 - \frac{4\mu}{B\Omega^2}\right] y = \ddot{y} + \left(\frac{\Omega}{2}\right)^2 \left[1 - \frac{1}{s}\right] y = 0$$

s if $\frac{\Omega^2}{4B\mu} = s$.

σ Designating $\sqrt{1 - 1/s}$ by σ, we finally get

(4.5) $$\ddot{y} + \left(\frac{\Omega}{2}\sigma\right)^2 y = 0 .$$

For convenience, following Kelley and McShane*, we rewrite (4.5) as

(4.6) $$\ddot{y} - q^2 y = \ddot{y} + (iq)^2 y = 0^{**}$$

and make another substitution

$$r = \frac{d \log y}{dt} = \frac{\dot{y}}{y} .$$

From this

$$\dot{y} = ry, \quad \ddot{y} = \dot{r}y + r\dot{y} = \dot{r}y + r^2 y$$

and (4.6) becomes

$$\dot{r}y + r^2 y - q^2 y = 0, \quad \text{from which}$$

(4.7) $$\dot{r} + r^2 - q^2 = 0 .$$

If $\dot{r}$ is small in comparison with r, we should have two solutions
q for r in this equation, one nearly equal to +q, and the other nearly equal to -q. Hence let

$$r = \pm q + \epsilon .$$

By substituting in (4.7), it appears that

$$\pm \dot{q} + \dot{\epsilon} + q^2 \pm 2q\epsilon + \epsilon^2 - q^2 = 0,$$

from which, if we neglect $\dot{\epsilon}$ and ϵ^2,

$$\epsilon = -\frac{\dot{q}}{2q}, \text{ and } r = \pm q - \frac{\dot{q}}{2q} .$$

*Kelley-McShane, BRL Report 446, 1944.

**Kelley-McShane refer to: H. Jeffreys, Proc. London Math. Soc. (2), vol. 23 (1923) p.428, and G. Wentzel, Zeitschrift für Physik, vol.38 (1926) p. 518.

From the definition of r,

$$r\,dt = d(\log y)$$

and

$$\int_0^t r\,dt = \log y - \log y_o \; .$$

From this, $\log y = \log y_o + \int_0^t r\,dt$

and $$y = y_o\, e^{\int_0^t r\,dt} = y_o\, e^{\int_0^t (\pm q - \frac{\dot{q}}{2q})dt} \; .$$

Thus, the two solutions for $+q$ and $-q$, respectively, are

$$y = y_1 \exp\left[\int_0^t (q - \frac{\dot{q}}{2q})dt\right]$$

$$y = y_2 \exp\left[\int_0^t (-q - \frac{\dot{q}}{2q})dt\right] .$$ Thus, y_1 and y_2 are constants.

From (4.5) and (4.6) we have

$$q^2 = (\frac{i\Omega\sigma}{2})^2$$

and $$+q = +\frac{i\Omega\sigma}{2}$$

$$-q = -\frac{i\Omega\sigma}{2} \; .$$

Furthermore

$$\frac{\dot{q}}{2q} = \frac{\frac{d(i\Omega\sigma)}{dt}}{2i\Omega\,\sigma} = \frac{1}{2}\left(\frac{\dot{\Omega}}{\Omega} + \frac{\dot{\sigma}}{\sigma}\right) .$$

Since A and B are constant,

$$\frac{\dot{\Omega}}{\Omega} = \frac{\dot{N}}{N} \quad \text{which we designate by } -\lambda ,$$

λ where λ is called the rolling moment damping factor. We have then,

substituting for q and $\dot{q}/2q$, for the two solutions

$$y = y_1 \exp \left[\frac{1}{2} \int (i\Omega\sigma + \ell - \frac{\dot{\sigma}}{\sigma}) \, dt \right]$$

$$y = y_2 \exp \left[\frac{1}{2} \int (- i\Omega\sigma + \ell - \frac{\dot{\sigma}}{\sigma}) \, dt \right] .$$

Recalling that $\eta = y \exp \left[\frac{1}{2} i \int_0^t \Omega dt \right]$ and that the general solution is the sum of the two solutions, we have

(5) $$\eta = K_1 e^{P_1} + K_2 e^{P_2}$$

where $$P_1 = \frac{1}{2} \int \left[i\Omega(1 + \sigma) + \ell - \frac{\dot{\sigma}}{\sigma} \right] dt$$

and $$P_2 = \frac{1}{2} \int \left[i\Omega(1 - \sigma) + \ell - \frac{\dot{\sigma}}{\sigma} \right] dt .$$

In (5), K_1 and K_2 are arbitrary complex constants determined by the initial conditions.

Interpretation of Solution

For those who have not had experience in the field of complex quantities, an interpretation of the solution (5) is offered.

Consider the series expansion of e^y.

$$e^y = 1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \text{etc.}$$

Upon substituting ix for y in this, we have

$$e^{ix} = 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} - \frac{x^6}{6!} + \ldots\ldots =$$

$$= \left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots \right] + i \left[x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots \right] .$$

The first term on the expansion of e^{ix} is the series for cos x while the

expression in the second bracket is the series for sin x. Hence we have

$$e^{ix} = \cos x + i \sin x \; .$$

For the case when $\mu \left[i \times A \right]$ is the only component of the force system and $\dot{\sigma} = \ell = 0$, the solution, as we have seen, is of the form

$$\eta = K_1 e^{\frac{1}{2} i\Omega(1 + \sigma)t} + K_2 e^{\frac{1}{2} i\Omega(1 - \sigma)t} \; .$$

Let us write this as

(6) $$\eta = K_1 e^{i\omega_1 t} + K_2 e^{i\omega_2 t}$$

and let us also assume for simplicity that K_1 and K_2 are both real.

Since m is the real part of η and n the imaginary part of η, then in view of the interpretation of e^{ix}, (6) may be written as:

(6.1) $$\begin{cases} m = K_1 \cos \omega_1 t + K_2 \cos \omega_2 t \\ n = K_1 \sin \omega_1 t + K_2 \sin \omega_2 t \end{cases}$$

Consider an epicyclic motion of the following character. Let there be a circle of radius K_2 as shown in Figure 3 and let a point P move on it with a constant angular velocity ω_2. With P as a moving center, let there be another circle of radius K_1 and a point, Q, moving on it with a constant angular velocity ω_1.

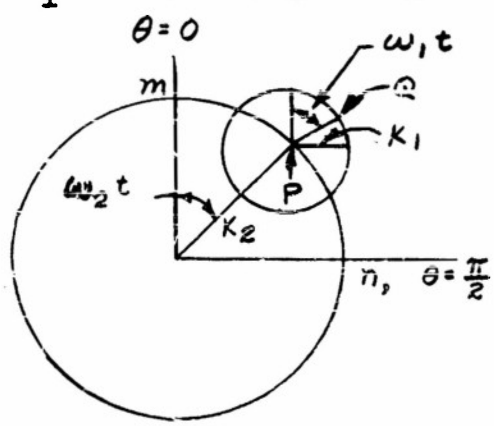

Figure 3

If $\theta = 0$ is the axis of m's and $\theta = \pi/2$ is the axis of n's, we have for m and n at any time t,

$$m = K_1 \cos \omega_1 t + K_2 \cos \omega_2 t$$

$$n = K_1 \sin \omega_1 t + K_2 \sin \omega_2 t .$$

But this is the same as given by (6.1) and the result establishes the epicyclic character of the motion given in (6).

In cases of practical interest, neither $\dot{\sigma}$ nor ℓ is zero. Let us rewrite P_1 and P_2 of (5) as

$$P_1 = i \frac{1}{2} \int \Omega (1 + \sigma)dt + \frac{1}{2} \int (\ell - \dot{\sigma}/\sigma) dt$$

$$P_2 = i \frac{1}{2} \int \Omega (1 - \sigma)dt + \frac{1}{2} \int (\ell - \dot{\sigma}/\sigma) dt .$$

The first parts of P_1 and P_2 are imaginary and the second parts are real. In view of the fact that σ is no longer taken as a constant, the angular velocities ω_1 and ω_2 are no longer constant. The real second parts indicate that the amplitudes are no longer either.

So, in place of (6.1), we write

$$m = \left\{ K_1 \cos \left[\frac{1}{2} \int \Omega(1+\sigma)dt \right] + K_2 \cos \left[\frac{1}{2} \int \Omega(1-\sigma)dt \right] \right\} \exp \left[\frac{1}{2} \int (\ell - \dot{\sigma}/\sigma)dt \right]$$

$$n = \left\{ K_1 \sin \left[\frac{1}{2} \int \Omega(1+\sigma)dt \right] + K_2 \sin \left[\frac{1}{2} \int \Omega(1-\sigma)dt \right] \right\} \exp \left[\frac{1}{2} \int (\ell - \dot{\sigma}/\sigma)dt \right] .$$

Chapter II

Equations of Motion with a Spinning Coordinate System. The Spins Caused by the Cross Wind Force, the Magnus Force and Gravity.

The Spinning Coordinate System

In the preceding, we have followed the methods of Fowler, Gallop, Lock, and Richmond in deriving equations (4.1) and (4.5) and the methods followed by Kelley and McShane[3] in obtaining the solution. In treating the motion in the presence of a cross wind force, and other forces transverse to the trajectory, we depart from Fowler, Gallop, Lock and Richmond, and adopt a coordinate system which is attached to the actual trajectory. As in Chapter I, the unit vector, i, points along the trajectory, the unit vector, j, is <u>initially</u> in a vertical plane through the trajectory, and the unit vector, k, is <u>initially</u> horizontal.*

ω

As a result of the cross wind force and other forces having components transverse to the trajectory, our coordinate system will have a component of spin, ω, about some axis transverse to the trajectory but no spin about the trajectory. By virtue of the spin, the coordinate system will not be a Galilean one. We proceed to derive first ω and then the equations of motion in this spinning non-Galilean coordinate system.

τ

Consider a body of mass, m, moving with velocity, v, in the direction indicated by the unit vector, τ. Its vector momentum is evidently $mv\tau$. The rate of change of momentum or the vector force is made up of two components, $m\dot{v}\tau$ and $mv\dot{\tau}$. The first component, parallel to τ, is the sum of the drag and the tangential component of the force of gravity, while the second component is a force perpendicular to τ. This is apparent since τ is a unit vector and the only possible change in τ is a change in direction. See Figure 4.

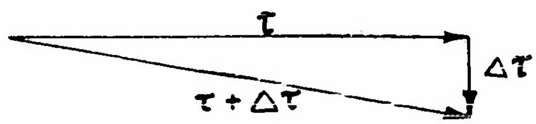

Figure 4

If we designate the vector force perpendicular to τ by F_n, we have

$$\dot{\tau} = \frac{F_n}{mv} \quad .$$

* The deviation of k from the horizontal direction and j from the vertical plane will be discussed in Chapter V. It will be shown that these deviations are small.

Thus, the angular velocity of the trajectory, ω, is equal in magnitude to

$$\frac{F_n}{mv} \; .$$

As a vector, it must be perpendicular to τ and also $\dot{\tau}$. Hence, the spin of our coordinate system, $\omega = \tau \times \dot{\tau} = \tau \times \frac{F_n}{mv}$.

It is apparent that $\frac{F_n}{mv}$ may be replaced by $\frac{F}{mv}$, where F is the resultant force on the shell, since the tangential component of F makes no contribution to the vector product.

In the future, we shall replace τ by i since by definition, i is the unit vector pointing along the trajectory. So that

(7) $$\omega = i \times \dot{i} = i \times \frac{F}{mv} \; .$$

From page 2, we have as the equation of motion

(1) $$A\dot{N}A + AN\dot{A} + B\left[\Lambda \times \ddot{\Lambda}\right] = G \; .$$

The superscript . indicates time derivatives in a Galilean coordinate system. If superscript primes indicate time derivatives in our chosen coordinate system, the job is to express $\dot{N}$, $\dot{\Lambda}$ and $\ddot{\Lambda}$ in terms of N', Λ', Λ'' and ω.

As the first step, we state the following theorem:

(7.1) $$\dot{r}_2 = r_1' + \omega \times r_1$$

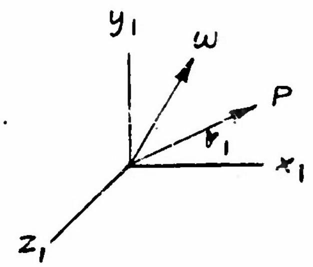

Figure 5

In this, r_1' is the rate of change of the vector, r, as measured in space 1 (non-Galilean) with axes x_1, y_1, z_1. Suppose this space is spinning with respect to Galilean space, 2, with an angular velocity, ω, as indicated. It is apparent that the velocity in space 2 of the point P at the end of r caused by the spin ω, will be $\omega \times r$. It may be

rigorously shown that the rate of change of r in space 2, $\dot{r}_2$*, is to be obtained by adding this term to r_1'. Hence, since space 2 is Galilean,

$$\dot{r}_2 = r_1' + \omega \times r_1 \; .^{**}$$

From this theorem, it follows directly that $\dot{\Lambda} = \Lambda' + \omega \times \Lambda$ and by a second application that

$$\ddot{\Lambda} = \Lambda'' + \omega \times \Lambda' + \omega' \times \Lambda + \omega \times \Lambda' + \omega \times \left[\omega \times \Lambda\right] =$$

$$= \Lambda'' + 2\,\omega \times \Lambda' + \omega' \times \Lambda + \omega \times \left[\omega \times \Lambda\right] \; .$$

Although ω has no component about the trajectory, it does have a component $\omega \sin \delta$ about the axis of the shell, which makes an angle δ with the tangent to the trajectory. We shall neglect $\omega \sin \delta$ in comparison with N. So we take the spin N unchanged in our coordinate system.

Upon making the substitutions indicated above in (1), we obtain

(1.02)
$$\begin{aligned} &AN'\Lambda + AN\Lambda' + AN\omega \times \Lambda \\ &+ B\Bigg\{\Lambda \times \Lambda'' + \overset{\text{I}}{\left[\Lambda \times \left[\omega' \times \Lambda\right]\right]} + 2\overset{\text{II}}{\left[\Lambda \times \left[\omega \times \Lambda'\right]\right]} \\ &+ \overset{\text{III}}{\left[\Lambda \times \left[\omega \times \left[\omega \times \Lambda\right]\right]\right]}\Bigg\} = G \; . \end{aligned}$$

To simplify this result, we make use of a theorem of vector analysis,

$$\left[A \times \left[B \times C\right]\right] = B\,(A\cdot C) - C\,(A\cdot B) \; .$$

For the two triple and one quadruple products indicated by the superscripts I, II and III, respectively, we obtain

$$\text{(I)} \quad \left[\Lambda \times \left[\omega' \times \Lambda\right]\right] = \omega'(\Lambda\cdot\Lambda) - \Lambda(\Lambda\cdot\omega') = \omega' - (\Lambda\cdot\omega')\Lambda$$

$$\text{(II)} \quad 2\left[\Lambda \times \left[\omega \times \Lambda'\right]\right] = 2\omega\,(\Lambda\cdot\Lambda') - 2\Lambda'\,(\Lambda\cdot\omega)$$

$$\text{(III)} \left[\Lambda \times \left[\omega \times \left[\omega \times \Lambda\right]\right]\right] = \omega(\Lambda\cdot\left[\omega \times \Lambda\right]) - \left[\omega \times \Lambda\right](\Lambda\cdot\omega) \; .$$

In treating the term $(\Lambda\cdot\omega')\Lambda$ of I, we consider two cases. The first case is the one where ω arises from aerodynamic forces and is therefore proportional to δ. Since in the final Λ, we consider only the j and k components which are themselves proportional to δ, it is apparent that,

*The use of the superscript dot is appropriate since space 2 is Galilean.

**The name Coriolis is associated with a similar transformation. See "Classical Mechanics", pp. 136-137, by Goldstein.

for this case, $(\Lambda\cdot\omega')\Lambda$ is at least of order $\delta\delta'$ and may be neglected in this first order theory. In the second case, ω arises from gravity and ω and ω' are always perpendicular to the particle trajectory; to the actual trajectory, they will be so nearly perpendicular that the product $(\Lambda\cdot\omega')$ will be of the order δ and $(\Lambda\cdot\omega')\Lambda$ of the order of δ^2. This also is neglected.

In II, since Λ is a unit vector, Λ' is necessarily perpendicular to Λ. Hence, the scalar product $(\Lambda\cdot\Lambda') = 0$. The spin ω is perpendicular to i, hence, the product $(\Lambda\cdot\omega)$ is at least of the order δ and Λ' is of the order δ'. The expression $-2\Lambda'(\Lambda\cdot\omega)$ is at least of the order $\delta\delta'$ and is thus omitted in this small yaw theory.

In III, the scalar product $(\Lambda\cdot[\omega \times \Lambda]) = 0$, since $[\omega \times \Lambda]$ is perpendicular to Λ.

In view of these considerations, (1.02) may be written

(1.03)
$$AN'\Lambda + AN\Lambda' + AN[\omega \times \Lambda] + B\left\{[\Lambda \times \Lambda''] + \omega' - [\omega \times \Lambda](\Lambda\cdot\omega)\right\} = G .$$

If the spin ω is caused by aerodynamic forces, ω will be proportional to δ and the term $[\omega \times \Lambda](\Lambda\cdot\omega)$ will be at least of the order δ^2 and thus may be neglected.

We proceed to compute the spins, ω_κ, ω_λ, and ω_g caused by the cross wind force, the Magnus force and gravity, respectively, thus obtaining the resultant spin $\omega = \omega_\kappa + \omega_\lambda + \omega_g$.

The Cross Wind Force

The cross wind force, L*, is defined to be a force which acts perpendicularly to the trajectory in the plane of the yaw. For this treatment, it is further assumed to be proportional to $\sin\delta$. So we take

$$L = \lambda \sin\delta,$$

and call λ the cross wind force co-factor.

Consider now the vector difference

$$(\Lambda - \cos\delta\, i); \text{ (see Figure 6).}$$

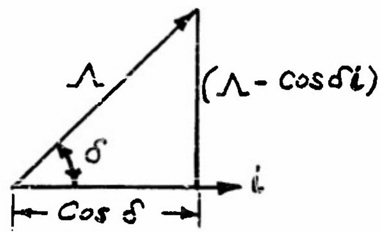

Figure 6

*In ballistics, we write

$$L = K_L\, \rho\, d^2 u^2 \sin\delta.$$ (See Hayes' Elements of Ordnance, p. 412).

It is obvious that $(\Lambda - \cos\delta\ \mathbf{i})$ has the magnitude $\sin\delta$ and that it is perpendicular to the trajectory, the direction of which is $\mathbf{i}$. Accordingly, a correct vector representation of the cross wind force is

$$\lambda\,(\Lambda - \cos\delta\cdot\mathbf{i})\ .$$

From (7) it appears that the spin ω_κ, of the coordinate system, caused by the cross wind force, is

$$\omega_\kappa = \mathbf{i} \times \frac{\lambda\,(\Lambda - \cos\delta\ \mathbf{i})}{mv} = \frac{\lambda}{mv}\left\{\mathbf{i}\times\Lambda - \cos\delta\left[\mathbf{i}\times\mathbf{i}\right]\right\} =$$

$$= \frac{\lambda}{mv}\left[\mathbf{i}\times\Lambda\right] = \frac{L}{mv\,\sin\delta}\left[\mathbf{i}\times\Lambda\right].$$

Upon replacing $\frac{\lambda}{mv}$ or $\frac{L}{mv\,\sin\delta}$ by κ, we have for the spin ω_κ caused by the cross wind force,

$$\omega_\kappa = \kappa\left[\mathbf{i}\times\Lambda\right].$$

Magnus Force

The Magnus force is a force which arises from the interaction of the boundary layer of a spinning shell and the wind stream when the shell has an angle of yaw. Consider a tennis ball or a baseball under the conditions shown in Figure 7. As a result of the interaction between the wind stream and the boundary layer, the velocity at the top surface of the ball will be

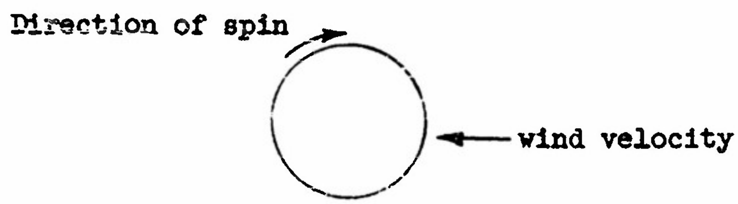

Figure 7

less than the velocity on the bottom surface of the ball. Associated with this will be a higher pressure over the top than there is on the bottom producing a force which accelerates the ball in a downward direction. This is what happens when a tennis ball is given a top spin.

Consider a shell spinning as shown in Figure 8. As a result of the

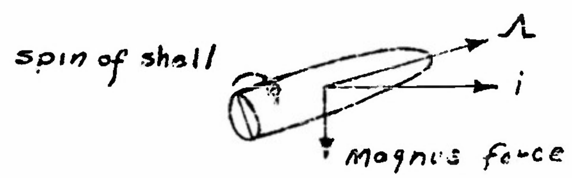

Figure 8

pressure distribution mentioned, there will be a Magnus force, K, acting perpendicular to the plane of the yaw, as indicated, which should be approximately proportional to the spin, N, the velocity of the shell, v, and to the sine of the angle of yaw, δ. If the factor of proportionality is represented by fm, we may write

K $$K^{*} = fmvN \sin \delta \quad \text{(in magnitude)}$$

and by

$$fmvN \left[\Lambda \times i\right] \text{ as a vector.}$$

λ We shall represent fN in the above by λ and call λ** the Magnus force damping factor. The contribution to $\dot{i}$ caused by K is therefore

$$\dot{i} = \frac{\lambda\, mv \left[\Lambda \times i\right]}{mv} = \lambda \left[\Lambda \times i\right] .$$

The spin ω_λ caused by the Magnus force is

$$\left[i \times \dot{i}\right] = \lambda \left[i \times \left[\Lambda \times i\right]\right] .$$

Gravity

g The force of gravity is numerically equal to mg where g is the acceleration caused by gravity. This force has a component perpendicular
θ to the trajectory equal in magnitude to mg cos θ, where θ is the inclination of the trajectory to the horizontal. Therefore, in our coordinate system, the component perpendicular to the trajectory will be represented by

$$- mg \cos \theta\, j^{***} .$$

It follows from (7) that the spin ω_g, caused by gravity, will be

$$i \times \frac{(-g \cos \theta\, j)}{v} = \theta' k$$

since $\theta' = \dfrac{- mg \cos \theta}{v}$.

The Resultant Spin of the Coordinate System

The spins caused by the cross wind force, the Magnus force and gravity are respectively,

$$\text{(8)} \quad \begin{cases} \omega_\kappa = \kappa \left[i \times \Lambda\right] , \\ \omega_\lambda = \lambda\, i \times \left[\Lambda \times i\right] , \\ \omega_g = \theta' k . \end{cases}$$

*In ballistic notation, we write, Magnus force=K=$K_K \rho d^2 Nv \sin \delta$(in magnitude).

**The damping factor λ is to be distinguished from the crosswind force cofactor λ.

***In view of the small rotation of our system about the trajectory, the component of the gravity force will not be exactly represented by - mg cos θj. There will, in general, be a small k component.

Chapter III

The Yawing Moment Due to Yawing.
The Magnus Moment.
The Equation for the Complex Yaw.

We have so far considered the following elements of the force system:

Drag
Overturning Couple
Axial Couple
Cross Wind Force
Magnus Force
Gravity

The last three forces contribute to the spin of the trajectory and, hence, to the spin of our coordinate system. In this chapter, we consider the yawing moment due to yawing and the Magnus moment, and derive the equation for the complex yaw.

Yawing moment due to yawing

H The yawing moment due to yawing, designated by Hw*, is the moment which opposes the angular velocity of the axis of the shell.

h We define a yawing moment due to yawing damping factor, h, by writing

$$Hw = - h\, Bw\ .$$

As we have seen in Chapter I (page 2), the angular velocity of the axis of the shell w may be expressed as $(A \times \dot{A})$. Thus, if we are to take account of this moment, we include on the right hand of equation (1.03) on page 14,

$$- h\, B\, (A \times \dot{A})$$

in addition to the overturning moment, $\mu \left[i \times A \right]$, and the axial couple, $-AN\Gamma A$.

As has been mentioned, $\dot{A}$ is the angular velocity referred to Galilean axes, whereas, in this problem, we refer the motion to axes turning with the trajectory. We thus have to express $\dot{A}$ in terms of A' and the resultant angular velocity of the trajectory, ω. As we have seen in Chapter II, $\dot{A}$ is equal to $A' + \omega \times A$, and thus

$$A \times \dot{A} = A \times A' + A \times \left[\omega \times A \right] .$$

As we have also seen in Chapter II, the triple product $A \times \left[\omega \times A \right]$ may be expressed as $\omega - (\omega \cdot A)A$. Hence

$$-hB \left[A \times \dot{A} \right] = -hB \left\{ \left[A \times A' \right] + \omega - (\omega \cdot A)A \right\} .$$

*In ballistic notation, Hw is expressed by

$$K_H\, \rho\, d^4\, vw$$

in which K_H is the yawing moment due to yawing coefficient and w is the vector angular velocity of the axis of the shell.

The term $(\omega \cdot \Lambda)\Lambda$ will be omitted since it is of the order δ^2 in the j and k components. Thus we have, including the moments so far considered, for the right hand side of (1.03),

$$\mu \left[i \times \Lambda \right] - AN \Gamma \Lambda - hB \left\{ \left[\Lambda \times \Lambda' \right] + \omega \right\} .$$

<u>Magnus Moment</u>

On page 15, it appeared that the vector Magnus force is proportional to $\Lambda \times i$. The moment of this force is perpendicular to the force and to the axis of the shell and is therefore properly given as proportional to $\Lambda \times \left[\Lambda \times i \right]$. We assume a proportionality factor of $AN\gamma$, where γ is the Magnus moment damping factor, and write the Magnus moment,

$$J^* = AN\gamma \left[\Lambda \times \left[\Lambda \times i \right] \right] .$$

This term is to be added to the right hand of equation (1.03) in addition to the moments mentioned above.

While it is commonly <u>assumed</u> in the small yaw theory that the Magnus force and moment are strictly proportional to the spin, there is little experimental evidence to back up this assumption. What evidence there is indicates that the force and moment are not exactly proportional to the spin. On the other hand, the evidence indicates that the Magnus force and moment are proportional to δ for small yaws.

Including all the moments so far considered, the right hand side of (1.03) is

$$\mu \left[i \times \Lambda \right] - AN \Gamma \Lambda - hB \left\{ \left[\Lambda \times \Lambda' \right] + \omega \right\} + AN\gamma \left[\Lambda \times \left[\Lambda \times i \right] \right] .$$

If the term $- AN \Gamma \Lambda$ is cancelled against $AN'\Lambda$, (1.03), see page 14, becomes

(1.04) $$AN(\Lambda' + \left[\omega \times \Lambda \right]) + B \left\{ \left[\Lambda \times \Lambda'' \right] + \omega' - (\Lambda \cdot \omega) \left[\omega \times \Lambda \right] \right\} =$$

$$= \mu \left[i \times \Lambda \right] - hB \left\{ \left[\Lambda \times \Lambda' \right] + \omega \right\} + AN\gamma \left[\Lambda \times \left[\Lambda \times i \right] \right]$$

with the spin, ω, as given in equation (7) of Chapter II by

$$\omega = \kappa \left[i \times \Lambda \right] + \lambda \left[i \times \left[\Lambda \times i \right] \right] + \theta' k .$$

We are to expand (1.04) in terms of the j and k components, the components of the complex yaw, and omit the i components, if any, since in this small yaw theory, $f = 1$.

---

*In ballistic notation, we write $J = K_J \rho \, d^2 \, N \, v \sin \delta$ (in magnitude).

Before carrying out this expansion, it is convenient to expand the vectors and the vector products in (1.04).

For ω, on carrying out the indicated vector multiplications, we find with

$$A = \ell\, i + mj + nk,$$

$$\omega = (-\kappa n + \lambda m)j + (\kappa m + \lambda n + \theta')k \,.$$

If the j and k components of the vectors and vector products in (1.04) are indicated by the notation $(\ )_{j,\,k,}$ we have

$$(A')_{j,\,k} = m'\,j + n'k$$

$$(\omega \times A)_{j,\,k} = (\kappa m + \lambda n + \theta')j + (\kappa n - \lambda m)k \,.$$

With the usual approximations,

$$(A \times A'')_{j,\,k} = -n''\,j + m''\,k$$

$$(\omega')_{j,\,k} = (-\kappa n' + \lambda m')j + (\kappa m' + \lambda n' + \theta'')k \,.$$

In treating the term $(\omega \cdot A)\left[\omega \times A\right]$ we need to include in ω only the term $\theta' k$ since ω will be proportional to δ for the other components and the term itself will be of the order δ^2 for these components.

With $\omega_g = \theta' k$,

$$\Big((\omega_g \cdot A)\left[\omega_g \times A\right]\Big)_{j,\,k} = n\theta'^2 j \,.$$

On the right hand side of (1.04),

$$\Big(\left[i \times A\right]\Big)_{j,\,k} = -nj + mk \,,$$

$$\Big(\left[A \times A'\right]\Big)_{j,\,k} = -n'j + m'k \,,$$

$$\Big(A \times \left[A \times i\right]\Big)_{j,\,k} = mj + nk \,.$$

Using these expressions for the vectors and vector products in (1.04), dividing both sides by B, expressing the quotient $\frac{AN}{B}$ by Ω, and collecting the terms in j and k, respectively, we obtain

$$\Omega(m' + \kappa m + \lambda n + \theta') - n'' + (-\kappa n' + \lambda m') - n\theta'^2 = -\frac{\mu}{B}n + hn' + h\kappa n - h\lambda m + \Omega\gamma m.$$

(1.05)

$$\Omega(n' + \kappa n - \lambda m) + m'' + (\kappa m' + \lambda n' + \theta'') = \frac{\mu}{B}m - hm' - h\kappa m - h\lambda n - h\theta' + \Omega\gamma n \,.$$

We multiply the first of these equations by $-i = -\sqrt{-1}$ and add to the second with the result:

$$\Omega\left[-im' + n' - \kappa(im - n) + \lambda(-m - in) - i\theta'\right]$$

$$+ m'' + in'' + \kappa(m' + in') - \lambda(im' - n') + in\theta'^2 + \theta'' =$$

$$= \frac{\mu}{B}(m + in) - h(m' + in') - h\kappa(m + in) + h\lambda(im - n) - h\theta' - \Omega\gamma(im - n) .$$

Replacing $(m + in)$ by η, the complex yaw, as in Chapter I, and arranging the terms, we have

$$\eta'' + \kappa\eta' - i\lambda\eta' - i\Omega\eta' - i\Omega\kappa\eta$$

$$-\Omega\lambda\eta - \frac{\mu}{B}\eta + h\eta' + h\kappa\eta - ih\lambda\eta + i\Omega\gamma\eta = i\Omega\theta' - \theta'' - in\theta'^2 - h\theta'$$

or

(1.06) $$\eta'' - i(\Omega + i\kappa + ih + \lambda)\eta' - \left[\frac{\mu}{B} + i\Omega\kappa + \Omega\lambda - i\Omega\gamma - h\kappa + ih\lambda\right]\eta =$$

$$= i\Omega\theta' - \theta'' - in\theta'^2 - h\theta' =$$

$$= i\Omega\left[\theta' - \frac{n\theta'^2}{\Omega} + i\frac{(\theta'' + h\theta')}{\Omega}\right].$$

Chapter IV

The Solutions of the Equation for the Complex Yaw, Stability.

In this chapter, we derive the solutions of the equation for the complex yaw (1.06). We first consider a particular solution of the equation. We insert the particular solution in the equation and subtract the resulting equation from (1.06). By so doing, we get an homogeneous equation.

The Particular Solution

We proceed with the derivation of the particular solution and shall consider later the solutions of the homogeneous equation.

Following Kelley and McShane, let us write (1.06) in the form

(1.07) $$\eta'' + a\eta' + b\eta = c \ .$$

Assume a solution of the form

$$\eta = \frac{c}{b} + \epsilon \ ;$$

on the assumption that the η'' and η' terms are relatively small. On inserting the solution in (1.07) and neglecting η'' and ϵ', one finds

$$\epsilon = -\frac{a}{b}\left(\frac{c}{b}\right)' ,$$

and

$$\eta = \frac{c}{b} - \frac{a}{b}\left(\frac{c}{b}\right)' ,$$

In terms of the notation of equation (1.06) this gives

(9) $$\eta = -\frac{iAN}{\mu}\left[\theta' - \frac{n\theta'^2}{\Omega} + \frac{i(\theta'' + h\theta')}{\Omega}\right] - \frac{iAN}{\mu}\left\{-\frac{iAN}{\mu}\left[\theta' - \frac{n\theta'^2}{\Omega} + \frac{i(\theta'' + h\theta')}{\Omega}\right]\right\}'$$

if all but the leading term in 'a' and 'b' of equation (1.07) are ignored*.

*Values of terms in 'a' and 'b' for the 37mm H.E. Shell M54 at a velocity of 2000 ft/sec and a twist of rifling of 1 in 30:

Ω	588.5 rad/sec
κ	1.39 sec^{-1}
h	5.51 sec^{-1}
λ	0.52 sec^{-1}
μ	1017 $lb.ft^2/sec^2$
μ/B	52,000 sec^{-2}
$\Omega\ \kappa$	819 sec^{-2}
γ	-2.06 sec^{-1}
$\Omega\ \gamma$	-1212 sec^{-2}

If $n\theta'^2$, θ'', and $h\theta'$ in the first bracket on the right are neglected and the second term, is neglected entirely, (9) becomes

$$\eta = \frac{-iAN}{\mu}\,\theta' = +\frac{iAN}{\mu}\,\frac{g\cos\theta}{v} \quad . \tag{9.1}$$

In view of the fact that the z axis is the axis of imaginaries, (9.1) indicates, on this approximation, that (a) the projectile is pointing directly to the right, and (b) the magnitude of the yaw is given by

$$\sin\delta = \frac{AN}{\mu}\,\frac{g\cos\theta}{v} \quad .$$

The preceding is therefore a derivation of the result given in Hayes, Chapter X, p. 420. Essentially, this expression for what is called the "yaw of repose" was used by Fowler, Gallop, Lock and Richmond in predicting the "right drift" of a projectile with a satisfactory agreement with experiment.

Solutions for the Homogeneous Equation

As was pointed out, if we subtract the special solution (9) from (1.06), we get the following homogeneous equation,

$$\eta'' - i(\Omega + i\kappa + ih + \lambda)\eta' - \left[\mu/B + i\Omega(\kappa - \gamma - i\lambda) - h\kappa + ih\lambda\right]\eta = 0 \quad . \tag{10}$$

As in Chapter I, we write (10) in the form

$$\eta'' - iE\eta' - F\eta = 0$$

and make the transformation

$$\eta = y \exp\left[\frac{1}{2}\, i \int E\, dt\right]$$

obtaining

$$y'' + \left(\frac{E^2}{4} - F + \frac{1}{2}\, i\, E'\right)y = 0 \quad .$$

Upon inserting the values of E and F from (10), this becomes

$$y'' + \left[\begin{array}{l} \frac{\Omega^2}{4} - \frac{\mu}{B} + \frac{\Omega}{2}(-i\kappa + ih + 2i\gamma - \lambda) + \frac{1}{2}h\kappa + \\ \frac{1}{2}i(\kappa\lambda + h\lambda) - \frac{1}{4}(\kappa^2 + h^2 - \lambda^2) + \\ \frac{1}{2}i(\Omega' + i\kappa' + ih' + \lambda') \end{array}\right] y = 0 \quad .$$

If we neglect the products and squares of the damping factors and also their derivatives and write

$$\Omega' = \frac{\Omega'}{\Omega}\,\Omega = -\ell\,\Omega \;,$$

the coefficient of y in (11) becomes

$$\left[\frac{\Omega^2}{4} - \frac{\mu}{B} + \frac{i\Omega}{2}(h - \kappa + 2\gamma - \ell) - \frac{\Omega}{2}\lambda\right].$$

In view of the fact that

$$\frac{\Omega^2}{4} - \frac{\mu}{B} = \left(\frac{\Omega}{2}\sigma\right)^2,$$

this may be written

$$\text{(12)} \qquad \left(\frac{\Omega}{2}\sigma\right)^2\left[1 + \frac{2i}{\Omega\sigma^2}(h - \kappa + 2\gamma - \ell) - \frac{2\lambda}{\Omega\sigma^2}\right].$$

This quantity (12) is $-q^2$ of equation (4.6), page 6. Upon taking the square root* of the bracket by the aid of the binomial theorem, i.e.,

$$\left[1 + x\right]^{\frac{1}{2}} = 1 + \frac{1}{2}x + \dots$$

and throwing away terms of higher degree than the first in x, we obtain

$$+q = +\frac{1}{2}i\left\{\Omega\sigma + i(h - \kappa + 2\gamma - \ell)/\sigma - \lambda/\sigma\right\}$$

$$-q = -\frac{1}{2}i\left\{\Omega\sigma + i(h - \kappa + 2\gamma - \ell)/\sigma - \lambda/\sigma\right\}.$$

In taking the derivative $\dot{q}$, we consider only the time derivatives of Ω and σ and neglect the others. Thus, as before (see page 7),

$$-\frac{\dot{q}}{2q} = \frac{1}{2}\left(\ell - \frac{\dot{\sigma}}{\sigma}\right).$$

*Following a suggestion of C.H. Murphy (BRL Technical Note 703), we may replace the bracket of (12) by

$$(P + iQ).$$

If now we place

$$a + ib = \sqrt{P + iQ},$$

$$a^2 - b^2 = P,\ 2ab = Q.$$

According to the approximation by the binomial theorem,

$$\sqrt{P + iQ} = \sqrt{P} + \frac{iQ}{2\sqrt{P}},$$

while, according to the above exact relations, the first term should not be $\sqrt{P}$ but approximately $\sqrt{P}\sqrt{1 + \frac{Q^2}{4P^2}}$. On the other hand, the second of these relations is fulfilled by the approximation of the binomial theorem.

Hence, the solution for y is

$$(13)\quad y = y_1 \exp\left[\frac{1}{2} i \int \left\{ \Omega\sigma - \frac{\lambda}{\sigma} + i\left[(h - \kappa + 2\gamma - \ell)/\sigma - \ell + \frac{\dot\sigma}{\sigma} \right] \right\} dt \right] +$$

$$y_2 \exp\left[-\frac{1}{2} i \int \left\{ \Omega\sigma - \frac{\lambda}{\sigma} + i\left[(h - \kappa + 2\gamma - \ell)/\sigma + \ell - \frac{\dot\sigma}{\sigma} \right] \right\} dt \right]$$

where y_1 and y_2 are complex constants.

If we write this as

$$y = y_1 e^{Q_1} + y_2 e^{Q_2},$$

then, in view of the fact that

$$\eta = y \exp\left[\frac{1}{2} i \int_0^t E\, dt\right]$$

and that

$$E = (\Omega + i\kappa + ih + \lambda),$$

we have for η,

$$(14)\quad \eta = K_1 \exp\left[\frac{1}{2} i \int_0^t (\Omega + ih + i\kappa + \lambda)\, dt + Q_1\right] +$$

$$K_2 \exp\left[\frac{1}{2} i \int_0^t (\Omega + ih + i\kappa + \lambda)\, dt + Q_2\right]$$

where K_1 and K_2 are complex constants.

Finally we write

$$(14.1)\quad \eta = K_1 e^{P_1} + K_2 e^{P_2}.$$

From (14) and (14.1), it is seen that

$$(15)\quad P_1 = \frac{1}{2}\int_0^t \left\{ i\left[\Omega(1 + \sigma) + \lambda - \lambda/\sigma\right] - \left[h + \kappa - \ell + \dot\sigma/\sigma + \overset{\mathrm{I}}{(h - \kappa + 2\gamma - \ell)/\sigma}\right] \right\} dt,$$

$$P_2 = \frac{1}{2}\int_0^t \left\{ i\left[\Omega(1 - \sigma) + \lambda + \lambda/\sigma\right] - \left[h + \kappa - \ell + \dot\sigma/\sigma - \overset{\mathrm{II}}{(h - \kappa + 2\gamma - \ell)/\sigma}\right] \right\} dt.$$

Since the stability factor is assumed to be appreciably greater than 1, the terms in λ are negligible compared to those in Ω*. Hereafter, λ will be

*E.g., for the 20mm Practice Projectile T114 fired at a muzzle velocity of 3026 ft/sec with a twist of rifling of one turn in 25.4 calibers,

$$\Omega = 1525 \text{ rad/sec},$$

$$\lambda = 0.2 \text{ sec}^{-1} \text{ (approximately)}.$$

neglected. Equations (15) then become

$$(15.1)\quad \begin{aligned} P_1 &= \frac{1}{2}\int_0^t \left\{ i\Omega(1+\sigma) - \left[h + \kappa - \ell + \dot{\sigma}/\sigma + (h - \kappa + 2\gamma - \ell)/\sigma \right] \right\} dt, \\ P_2 &= \frac{1}{2}\int_0^t \left\{ i\Omega(1-\sigma) - \left[h + \kappa - \ell + \dot{\sigma}/\sigma - (h - \kappa + 2\gamma - \ell)/\sigma \right] \right\} dt . \end{aligned}$$

The General Solution

If the particular solution (9) is represented by $c + id$ where c and d are both real, then the general solution (the sum of the particular solution and the solutions of the homogeneous equation) is

$$\eta = K_1 e^{P_1} + K_2 e^{P_2} + c + id .$$

If the initial value of η is η_o, etc., the constants K_1 and K_2 must satisfy the conditions:

$$\eta_o = K_1 + K_2 + c_o + id_o$$

$$\eta_o' = K_1 P_{1,0}' + K_2 P_{2,0}' + c_o' + id_o' .$$

Stability conditions for spinning shell

As we have seen in Chapter I, the imaginary parts of P_1 and P_2 determine the frequency or angular velocity of the motion while the real parts control the amplitude.

A shell is ordinarily said to be dynamically stable if the amplitudes of both motions diminish with time. Thus, obviously, the conditions for this sort of dynamic stability are that the real parts of P_1 and P_2 must be negative. However, in such a treatment of the dynamical stability of shell, no attention is paid to the variation of these factors with velocity and, as a rule, the factors vary considerably as the velocity changes. It will be noticed that the amplitude depends not on the integrands involved in P_1 and P_2 but on the integrals. Therefore, the integrand may be positive over certain parts of the velocity interval considered while the integral may be negative. It is apparent that the integrand itself should be negagive at the muzzle velocity at which the shell is fired because, if it is not, the large amplitudes might develop before the integral becomes negative once more. Thus, practically considered, there are two factors that have to be considered in designing a shell: (1) the integrand of the real parts in P_1 and P_2 should be negative in the neighborhood of the muzzle velocity, but,(2)

on the other hand, if the integrands are reasonably negative in the neighborhood of the muzzle velocity, it is not essential that they be negative at all other velocities which the shell assumes on its trajectory. It is known that there are quite a number of shell which, in the narrow sense of the word, are dynamically unstable on certain parts of their trajectories but nevertheless are exceedingly accurate. It would be throwing away many perfectly good shell designs if one insisted that the integrand be negative at all velocities which the shell takes on during its trajectory.

If σ is real, the four conditions for the satisfactory damping of the yaw of a shell on a given trajectory are:

near the muzzle

$$(a)\quad \left[h + \kappa - \ell + \dot{\sigma}/\sigma + (h - \kappa + 2\gamma - \ell)/\sigma \right] > 0$$

(dynamically stable)

$$(b)\quad \left[h + \kappa - \ell + \dot{\sigma}/\sigma - (h - \kappa + 2\gamma - \ell)/\sigma \right] > 0$$

along the trajectory

$$(c)\quad \int_0^t \left[h + \kappa - \ell + \dot{\sigma}/\sigma + (h - \kappa + 2\gamma - \ell)/\sigma \right] dt > 0$$

(not necessarily dynamically stable at all points on the trajectory)

$$(d)\quad \int_0^t \left[h + \kappa - \ell + \dot{\sigma}/\sigma - (h - \kappa + 2\gamma - \ell)/\sigma \right] dt > 0\ ,$$

when t is any time along the trajectory*.

As Murphy suggests in BRL Report 853, for practical purposes, the 0's on the right of the above inequalities might well be replaced by positive constants.

*As Sterne and Poor have pointed out, it may be dangerous to design a weapon for which all four stability conditions are satisfied only when the shell is fired for a certain muzzle velocity. If the muzzle velocity were inadvertently changed, the conditions 'a' and 'b' might no longer be satisfied.

Chapter V

The Motion of the Coordinate System.
Comparison with Fowler, Gallop, Lock and Richmond.
Comparison with Kelley and McShane.
Neglected Forces.
Acknowledgments.

The Motion of the Coordinate System

As compared with the coordinate system of Fowler, Gallop, Lock and Richmond, which is attached to the particle trajectory, ours has the advantage that it deals with only one vector, A, instead of two, A and X (see below). As compared with that of Kelley and McShane*, it has the advantage of resembling more closely a Galilean system than one whose coordinates oscillate with the projectile. However, our coordinate system has the disadvantage that the z axis, originally assumed to be horizontal, will not, in general, remain horizontal, and the y axis, originally in a vertical plane through the trajectory, will not, in general, remain in that plane. However, it will be shown that the departure of the z axis from the horizontal will be small and the departure of the y axis from the vertical plane will be also.

Consider a projectile moving vertically upward. Let us take the y axis initially as coming out of the plane

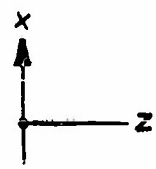

Figure 9

of Figure 9 and the z axis as pointing to the right. Suppose there is a spin ω_y of the coordinate system about the y axis, then the z axis will turn toward the vertical with the same angular velocity ω_y. Place a vertical plane so that it will include the z axis and let ϕ be the angle, measured in that plane between a horizontal line and the z axis. In this case, $\dot{\phi} = \omega_y$.

Let us consider the general case. Suppose the tangent to the trajectory makes an angle θ with the horizontal, let the z axis point initially directly to the right and let the y axis be initially in the vertical plane thru the x axis. See Figure 10.

*Kelley and McShane have adopted the coordinate system of Nielsen and Synge. Cf. "On the Motion of a Spinning Shell," Quarterly of Applied Mathematics, Vol. IV, No. 3, October 1946, pp. 201-226.

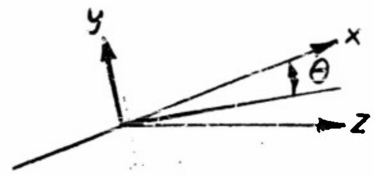

Figure 10

It appears that the component of ω_y about an horizontal normal to the initial direction of the z axis will be $\omega_y \sin\theta$. So, in general,

$$\dot\phi = \omega_y \sin\theta \ .$$

It is apparent that a rotation ω_z about the z axis, if z is horizontal, will not move the y axis from the vertical.

On the other hand, if the z axis is not horizontal, a rotation about it, ω_z, will cause the y axis to leave the vertical plane through the trajectory. However, $\dot\phi$ is approximately proportional to ω_y as long as $\dot\phi$ is small. Thus, the departure of the y axis from the vertical will be approximately proportional to the product $\omega_y \cdot \omega_z$ and will be neglected.

It is obvious that a rotation about the y axis will not change the direction of the y axis.

It thus appears that the rate of departure of the z axis from the horizontal is given by $\dot\phi = \omega_y \sin\theta$, and that the rate of departure of the y axis will be of higher order in the spins of the trajectory.

The spin of the coordinate system caused by gravity, ω_g, has no component in ω_y. The main component of ω_y is that caused by the cross wind force, in magnitude, $-\kappa\eta$.

With an assumed value of $\kappa = 1.0$, Hitchcock and Harrington have computed

$$\frac{\phi}{\sin\theta} = \frac{\int \dot\phi \, dt}{\sin\theta} = \frac{\int \omega_y \, dt}{\sin\theta}$$

for two cases, one where the motion is undamped (with damping due to κ neglected) and one where the motion is damped with $h = 3.2$ and $\kappa \doteq 1.0$. This was for a 3.3" shell with a velocity of 2080 ft/sec. The results were that, at the end of .6 sec., the maximum value of $\frac{\phi}{\sin\theta}$ for the undamped case was less than .003 rad, and less than .002 rad for the damped case.

Comparison with Results of FGLR

Fowler and associates refer the motion to a coordinate system attached to the particle trajectory with only drag and gravity acting on the particle. In this system, they consider two vectors. One which we shall call Λ_F is the unit vector having the direction of the axis of the shell; the other, X, a unit vector having the direction of the actual trajectory, both referred to the coordinate system mentioned.

Fowler writes

$$\Lambda_F = \ell_F i_F + m_F j_F + n_F k_F$$

$$X = x i_F + y j_F + z k_F \, . \qquad (x,y,z \text{ are direction cosines.})$$

If we define the vector yaw indicated by Δ as the component of the vector Λ_F normal to the actual trajectory, then with the coordinate system of FGLR,

$$\Delta = \Lambda_F - X = (m_F - y) j_F + (n_F - z) k_F$$

since $\ell \approx x \approx 1$.

On the other hand, the vector representation of our complex yaw is $(mj + nk)$. Hence, if the small angles between our and their coordinate axes may be disregarded, i.e., if $j_F = j$, $k_F = k$; $m = m_F - y$, $n = n_F - z$.

FGLR introduce two complex variables in their treatment, η and ζ . The definitions of these variables are as follows:

$$\eta + c\,\zeta = m_F + i n_F \, , \quad c\,\zeta = y + iz$$

where $c = \cos\theta$.

From these, it is seen that

$$\eta = (m_F - y) + i(n_F - z) \, ,$$

and Fowler's η is equivalent to our η, except for the slight difference between our axes and his.

On page 337, FGLR give, with slight changes in notation, the following equations involving η and ζ .

$$(3.613) \quad \eta'' - (i\Omega - h - \kappa + i\lambda)\eta' - \left\{ \frac{\Omega^2}{4s} + i\Omega(\kappa - i\lambda - \gamma) - h\kappa + ih\lambda - \kappa' + i\lambda' - (\kappa - i\lambda)\frac{c'}{c} \right\} \eta$$

$$- \left[i\Omega c' - hc' - c'' \right] \zeta = i\Omega\ast = i\Omega(\theta_1' + i\theta_1''/\Omega)^* \, .$$

* Where θ_1 is the inclination of the particle trajectory.

This is to be compared with our equation (1.06) on page 20. The differences are as follows:

a. FGLR have two terms in η: $-(\kappa' - i\lambda')\eta$ and $+ (\kappa - i\lambda) \frac{c'}{c} \eta$.

b. They have a ζ term which they later discard as being of the order $\frac{1}{\Omega^2}$.

c. On the right hand side, we have two terms in the bracket of (1.06)

$$\frac{n\theta'^2}{\Omega} \quad \text{and} \quad \frac{ih\theta'}{\Omega}$$

which Fowler considers negligible.

FGLR give as the solution (p. 339) for the simplified homogeneous equation in η, with a slight change of form,

$$\eta = \left(\frac{\Omega\sigma}{\Omega_o\sigma_o}\right)^{-1/2} \left[K_1 e^{P_1} + K_2 e^{P_2}\right] \tag{3.6234}$$

where

$$P_1, P_2 = \frac{1}{2} \int_0^t \left\{ i\Omega(1 \pm \sigma) - \left[h + \kappa \pm (h - \kappa + 2\gamma - \ell)/\sigma\right] \right\} dt$$

and K_1 and K_2 are arbitrary constants.

It may be shown that

$$\left(\frac{\Omega\,\sigma}{\Omega_o\sigma_o}\right)^{-\frac{1}{2}} = \exp\left[\frac{1}{2} \int (\ell - \frac{\dot{\sigma}}{\sigma})\, dt\right].$$

In view of this, (3.6234) may be written

$$\eta = K_1 e^{P_1} + K_2 e^{P_2}$$

with the P's now defined by

$$P_1, P_2 = \frac{1}{2} \int \left\{ i\Omega(1 \pm \sigma) - \left[h + \kappa - \ell + \frac{\dot{\sigma}}{\sigma} \pm (h - \kappa + 2\gamma - \ell)/\sigma\right] \right\} dt .$$

This is to be compared with (15.1) on page 25. It will be seen that the results are identical.

Another comment on Fowler's treatment: he claims his result to be valid only for large values of Ω; our results indicate that his results are of greater generality than he realized. In fact, they merely depend

on the usual W.K.B. approximations (see p. 6)*.

Comparison with Kelley-McShane

Kelley-McShane use the arc length as the independent variable, instead of the time. Dr. Galbraith has kindly transformed Kelley-McShane's equation (1.23), BRL Report 446, to its equivalent with time as the independent variable. The result, with our notation for the damping factors, is:

$$\ddot{\eta} - i\dot{\eta}(\Omega + i\kappa + ih + \lambda + \psi) - \eta\left\{\frac{\mu}{B} + i\Omega(\kappa - \gamma - i\lambda) - i\,\frac{\Omega g \sin\theta}{u}\right\}$$

$$= i\Omega\,\theta'\left(1 + \frac{\psi}{\Omega} + \frac{ih}{\Omega}\right) - \theta'' .$$

*In view of the fact that FGLR made their observations by yaw cards, they give, in addition to an expression for η, the complex yaw, expressions for δ^2 and for ϕ, the angle between a vertical plane through the trajectory and the plane of the yaw:

$$\delta^2 = \alpha^2 \sin^2 qt + \beta^2 \cos^2 qt ,$$

$$\phi = \phi_o + \frac{1}{2}\Omega t + \arctan\left\{\coth(j - q_3)\tan qt\right\} .$$

In these,

α = maximum yaw

β = minimum yaw

$$q = \frac{1}{2}\Omega\sigma$$

$$\alpha = Je^{-q_1}\cosh(q_2 - j)$$

$$\beta = Je^{-q_1}\sinh(q_2 - j)$$

$$\tanh j = \beta_o/\alpha_o$$

$$J = \alpha_o/\cosh j = \beta_o/\sinh j$$

$$q_1 = \int_0^t \frac{h + \kappa}{2}\,dt$$

$$q_2 = \int_0^t \frac{h - \kappa + 2\gamma - \ell + \dot{\sigma}/\sigma}{2\sigma}\,dt$$

$$q_3 = \int_0^t \frac{h - \kappa + 2\gamma - \ell}{2\sigma}\,dt .$$

If this is compared with (1.06) on page 20, it is apparent that, aside from the terms in ψ which we neglect, the only difference is the term $\frac{i\Omega g \sin \theta}{u}$ in the coefficient of η. In view of the fact that this term is omitted in "Exterior Ballistics", by McShane, Kelley and Reno, it is apparent that Kelley and McShane consider it should be omitted.

Neglected Forces

We have so far neglected three elements of the force system because, in practice, the corresponding damping factors are negligible. These are:

the force due to yawing,

the Magnus force due to yawing,

the Magnus moment due to yawing.

a. The force due to yawing

The yawing moment due to yawing is produced by the product of a force multiplied into the distance between the center of application of the force and the center of gravity. This force is the force due to yawing. It is important in the motion of dirigibles and of interest in developing a theory of the yawing moment due to yawing. It has a negligible effect on the motion of ordinary shell.

b. The Magnus force due to yawing

Just as there is a Magnus force which corresponds to the cross wind force, so there is also a Magnus force due to yawing. As a result of the spin of a shell, the Magnus force due to yawing differs in magnitude and direction from the force due to yawing. This force is also negligibly small and need not be considered in the theory of damping.

c. The Magnus moment due to yawing

Corresponding to the yawing moment due to yawing, if the shell is spun, there will be a Magnus moment due to yawing but this Magnus moment is also negligibly small.

Acknowledgments

I have received much helpful advice and many useful suggestions in the preparation of these notes: from Professor Birkhoff - much sound advice on clarity and rigor; from Dr. Galbraith - valuable help in comparing this theory with that of Kelley and McShane; and from Mr. Hitchcock - much careful reading of the manuscript. Mr. Hitchcock and Miss M. E. Harrington also computed the motion of my coordinate system to determine the departure from the horizontal of an axis originally horizontal.

R. H. Kent

R. H. KENT

DISTRIBUTION

No. of Copies	Organization
6	Chief of Ordnance Department of the Army Washington 25, D. C. Attn: ORDTB - Bal Sec
10	British - ORDTB for distribution
4	Canadian Joint Staff - ORDTB for distribution
4	Chief, Bureau of Ordnance Department of the Navy Washington 25, D. C. Attn: Re3
4	ASTIA Reference Center Library of Congress Washington 25, D. C.
3	Commander Naval Ordnance Laboratory White Oak Silver Spring, Maryland
3	Commander Naval Ordnance Test Station Inyokern, China Lake, California Attn: Technical Library
1	Superintendent Naval Postgraduate School Monterey, California
1	Commander Naval Air Development Center Johnsville, Pennsylvania
1	Commander Naval Air Missile Test Center Point Mugu, California
5	Director National Advisory Committee for Aeronautics 1724 F Street, N.W. Washington 25, D. C. Attn: Division Research Information
2	Director National Advisory Committee for Aeronautics Ames Laboratory Moffett Field, California Attn: Dr. A. C. Charters Mr. H. J. Allen
1	Commanding Officer and Director David W. Taylor Model Basin Washington 7, D. C. Attn: Aerodynamics Laboratory
1	Chief of Staff U. S. Air Force The Pentagon Washington 25, D. C. Attn: DCS/D, AFDRD-AC-3
1	Chief, Armed Forces Special Weapons Project P. O. Box 2610 Washington 25, D. C.
5	Director Armed Services Technical Information Agency Documents Service Center Knott Building Dayton 2, Ohio Attn: DSC - SA
3	Commanding General Redstone Arsenal Huntsville, Alabama
3	Commanding General Picatinny Arsenal Dover, New Jersey
3	Commanding Officer Frankford Arsenal Philadelphia 37, Pennsylvania
1	Commanding Officer Camp Dietrick Frederick, Maryland

DISTRIBUTION

No. of Copies	
1	Professor of Ordnance U. S. Military Academy West Point, New York
1	Professor G. Birkhoff Department of Mathematics Harvard University 2 Divinity Avenue Cambridge 38, Massachusetts
1	Professor George F. Carrier Division of Applied Science Harvard University Cambridge 38, Massachusetts
1	Professor Francis H. Clauser Department of Aeronautics Johns Hopkins University Baltimore 18, Maryland
1	Professor C. S. Draper Department of Aeronautical Engineering Massachusetts Institute of Technology Cambridge 39, Massachusetts
1	Professor F. W. Loomis Head, Department of Physics University of Illinois Urbana, Illinois
1	Professor E. J. McShane Department of Mathematics University of Virginia Charlottesville, Virginia
1	Professor Clark B. Millikan Guggenheim Aeronautical Laboratory California Institute of Technology Pasadena 4, California

No. of Copies	
1	Dr. Allen E. Puckett Hughes Aircraft Company Florence Avenue at Teal Street Culver City, California
1	Prof. N. F. Ramsey, Jr. Department of Physics Harvard University Cambridge 38, Massachusetts
1	Dr. L. H. Thomas Watson Scientific Computing Laboratory 612 West 116th Street New York 27, New York
1	Prof. John von Neumann Institute for Advanced Study Princeton, New Jersey
1	Dr. R. E. Bolz Case Institute of Technology University Circle Cleveland, Ohio
1	Professor J. W. Cell North Carolina State College Raleigh, North Carolina
1	Dr. J. L. Kelley Department of Mathematics University of California Berkeley, California
1	Dr. Z. Kopal Department of Electrical Engineering Massachusetts Institute of Technology Cambridge 39, Massachusetts

DISTRIBUTION

No. of Copies	Organization
1	Professor G. Kuerti Department of Mechanical Engineering Case Institute of Technology Cleveland, Ohio
2	Armour Research Foundation Illinois Institute of Technology Technology Center Chicago 16, Illinois Attn: Mr. W. Casler Dr. A. Wundheiler
2	Applied Physics Laboratory Johns Hopkins University Silver Spring, Maryland
1	Dr. B. H. Sage Department of Chemical Engineering California Institute of Technology Pasadena, California
2	North American Aviation, Inc. 12214 Lakewood Boulevard Downey, California Attn: Aerophysics Library
1	Northrop Aircraft, Inc. Hawthorne, California
2	Massachusetts Institute of Technology Cambridge 39, Massachusetts Attn: Guided Missiles Library Room 22-001
2	Sandia Corporation Sandia Base P. O. Box 5800 Albuquerque, New Mexico Attn: Mrs. Wynne K. Cox
1	Professor J. W. Beams Chairman, Department of Physics University of Virginia Charlottesville, Virginia

No. of Copies	Organization
1	Dr. P. J. Gingo Defense Research Division Firestone Tire and Rubber Co. Akron, Ohio
1	Dr. Leverett Davis Norman Bridge Laboratory of Physics California Institute of Technology Pasadena, California
1	Colonel H. U. Wagner 1822 Wyoming Avenue Washington 9, D. C.